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Extension theorems concerned with results by Ponnusamy and Karunakaran

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1 Introduction

Let $\mathcal{A}(n,k)$ be the class of functions $f(z)$ of the form

$$f(z)=z^n+\sum_{m=n+k}^{\infty}a_m z^m \qquad (n \geqq 1, k \geqq 1) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z \in \mathbb{C}:|z|<1\}$. For two functions $f(z)$ and $g(z)$ belonging to the class $\mathcal{A}(1,1)$, Sakaguchi [5] has proved the following result.

Theorem A *Let $f(z) \in \mathcal{A}(1,1)$ and $g(z) \in \mathcal{A}(1,1)$ be starlike in* $\mathbb{U}$. *If $f(z)$ and $g(z)$ satisfy*

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right)>0 \qquad (z \in \mathbb{U}), \tag{1.2}$$

then

$$\mathrm{Re}\left(\frac{f(z)}{g(z)}\right)>0 \qquad (z \in \mathbb{U}). \tag{1.3}$$

After Theorem A, many mathematicians studying this field have applied this theorem to get some results. In 1989, Ponnusamy and Karunakaran [4] have improved Theorem A as following.

Theorem B *Let α be a complex number with $\mathrm{Re}\,\alpha>0$ and $\beta<1$. Further, let $f(z) \in \mathcal{A}(n,k)$ and $g(z) \in \mathcal{A}(n,j)$ $(j \geqq 1)$ satisfies*

$$\mathrm{Re}\left(\frac{\alpha g(z)}{zg'(z)}\right)>\delta \qquad (z \in \mathbb{U}) \tag{1.4}$$

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with $0 \leqq \delta < \dfrac{\operatorname{Re}\alpha}{n}$. *If* $f(z)$ *and* $g(z)$ *satisfy*

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \beta \qquad (z \in \mathbb{U}), \tag{1.5}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > \frac{2\beta + \delta k}{2 + \delta k} \qquad (z \in \mathbb{U}). \tag{1.6}$$

It is the purpose of the present paper is to discuss Theorem B applying the lemma due to Fukui and Sakaguchi [1]. To discuss our problems, we need the following lemmas.

Lemma 1 *Let* $w(z) = \sum_{n=k}^{\infty} a_n z^n \quad (a_k \neq 0, k \geqq 1)$ *be analytic in* $\mathbb{U}$. *If the maximum value of* $|w(z)|$ *on the circle* $|z| = r < 1$ *is attained at* $z = z_0$, *then we have*

$$\frac{z_0 w'(z_0)}{w(z_0)} = \ell \geqq k, \tag{1.7}$$

which shows that $\dfrac{z_0 w'(z_0)}{w(z_0)}$ *is a positive real number.*

The proof of Lemma 1 can be found in [1] and we see that Lemma 1 is a generalization of Jack's lemma given by Jack [2]. Applying Lemma 1, we derive

Lemma 2 *Let* $p(z) = 1 + \sum_{n=k}^{\infty} c_n z^n \quad (c_k \neq 0, k \geqq 1)$ *be analytic in* $\mathbb{U}$ *with* $p(z) \neq 0$ $(z \in \mathbb{U})$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\operatorname{Re} p(z) > 0 \qquad (|z| < |z_0|)$$

and

$$\operatorname{Re} p(z_0) = 0,$$

then we have

$$-z_0 p'(z_0) \geqq \frac{\ell}{2}(1 + |p(z_0)|^2) \tag{1.8}$$

and so

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell, \tag{1.9}$$

where

$$k \leqq \frac{k}{2}\left(a + \frac{1}{a}\right) \leqq \ell \qquad \left(\arg p(z_0) = \frac{\pi}{2}\right) \tag{1.10}$$

and

$$-k \geqq -\frac{k}{2}\left(a + \frac{1}{a}\right) \geqq \ell \qquad \left(\arg p(z_0) = -\frac{\pi}{2}\right) \tag{1.11}$$

with $p(z_0) = \pm ia \quad (a > 0)$.

Proof. Let us consider

$$\phi(z)=\frac{1-p(z)}{1+p(z)}=\frac{c_k}{2}z^k+\cdots \tag{1.12}$$

for $p(z)$. Then, it follows that $\phi(0)=\phi'(0)=\cdots=\phi^{(k-1)}(0)=0$, $|\phi(z)|<1 \quad (|z|<|z_0|)$ and $|\phi(z_0)|=1$. Therefore, applying Lemma 1, we have that

$$\frac{z_0\phi'(z_0)}{\phi(z_0)}=\frac{-2z_0p'(z_0)}{1-\left(p(z_0)\right)^2}=\frac{-2z_0p'(z_0)}{1+|p(z_0)|^2}=\ell\geqq k. \tag{1.13}$$

This implies that $z_0p'(z_0)$ is a negative real number and

$$-z_0p'(z_0)\geqq\frac{k}{2}(1+|p(z_0)|^2). \tag{1.14}$$

Let us use the same method by Nunokawa [3]. If $\arg p(z_0)=\frac{\pi}{2}$, then we write $p(z_0)=ia$ $(a>0)$. This gives us that

$$\operatorname{Im}\left(\frac{z_0p'(z_0)}{p(z_0)}\right)=\operatorname{Im}\left(-\frac{iz_0p'(z_0)}{a}\right)\geqq\frac{k}{2}\left(a+\frac{1}{a}\right).$$

If $\arg p(z_0)=-\frac{\pi}{2}$, then we write $p(z_0)=-ia \quad (a>0)$. Thus we have that

$$\operatorname{Im}\left(\frac{z_0p'(z_0)}{p(z_0)}\right)=\operatorname{Im}\left(\frac{iz_0p'(z_0)}{a}\right)\leqq-\frac{k}{2}\left(a+\frac{1}{a}\right).$$

This completes the proof of Lemma 2. □

2 Main results

With the help of Lemma 2, we derive the following theorem.

Theorem 1 *Let α be a complex number with $\operatorname{Re}\alpha>0$ and $\beta<1$. Further, let $f(z)\in\mathcal{A}(n,k)$ and $g(z)\in\mathcal{A}(n,j) \quad (j\geqq 1)$ satisfies*

$$\operatorname{Re}\left(\frac{\alpha g(z)}{zg'(z)}\right)>\delta \qquad (z\in\mathbb{U}) \tag{2.1}$$

with $0\leqq\delta<\frac{\operatorname{Re}\alpha}{n}$. If $f(z)$ and $g(z)$ satisfy

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}\right\}+\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z)}{g(z)}-\beta_1\right|^2>\beta \qquad (z\in\mathbb{U}) \tag{2.2}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)>\beta_1 \qquad (z\in\mathbb{U}), \tag{2.3}$$

where $\beta_1=\frac{2\beta+\delta k}{2+\delta k}$.

Proof. Defining the function $p(z)$ by

$$p(z)=\frac{\dfrac{f(z)}{g(z)}-\beta_1}{1-\beta_1}, \tag{2.4}$$

we see that $p(0)=1$ and

$$\begin{aligned}(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}-\beta \\ &= (\beta_1-\beta)+(1-\beta_1)\left(p(z)+\frac{\alpha g(z)}{zg'(z)}zp'(z)\right) \\ &> -\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z)}{g(z)}-\beta_1\right|^2\end{aligned} \tag{2.5}$$

for all $z\in\mathbb{U}$. Let us suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$|\arg p(z_0)|<\frac{\pi}{2}\qquad (|z|<|z_0|)$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}.$$

Then, by means of Lemma 2, we have that

$$-z_0p'(z_0)\geqq\frac{k}{2}(1+|p(z_0)|^2). \tag{2.6}$$

If follows from the above that

$$\begin{aligned}\mathrm{Re}\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)}+\alpha\frac{f'(z_0)}{g'(z_0)}-\beta\right\} \\ &= (\beta_1-\beta)+(1-\beta_1)\mathrm{Re}\left\{p(z_0)+\frac{\alpha g(z_0)}{z_0g'(z_0)}z_0p'(z_0)\right\} \\ &= (\beta_1-\beta)-(1-\beta_1)\mathrm{Re}\left\{\frac{\alpha g(z_0)}{z_0g'(z_0)}(-z_0p'(z_0))\right\} \\ &\leqq (\beta_1-\beta)-(1-\beta_1)\frac{\delta k}{2}(1+|p(z_0)|^2) \\ &= -\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z_0)}{g(z_0)}-\beta_1\right|^2\end{aligned}$$

which contradicts (2.5). This completes the proof of the theorem. □

Remark 1 If $f(z)$ and $g(z)$ satisfy $f(z_0) = \beta_1 g(z_0)$ in Theorem 1, then Theorem 1 becomes Theorem B given by Ponnusamy and Karunakaran [4]. We also have

Theorem 2 *Let α be a complex number with $\operatorname{Re}\alpha > 0$ and $\beta < 1$. Further, let $f(z) \in \mathcal{A}(n,k)$ and $g(z) \in \mathcal{A}(n,j)$ $(j \geqq 1)$ satisfies the condition (2.1) with $0 \leqq \delta < \dfrac{\operatorname{Re}\alpha}{n}$. If $f(z)$ and $g(z)$ satisfy*

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta\right\}\right| < \frac{\pi}{2} + \operatorname{Tan}^{-1}\left(\frac{\delta k|p(z)|}{2\left(\dfrac{2r}{1-r^2} + \dfrac{|\operatorname{Im}\alpha|}{n} + 1\right)}\right) \tag{2.7}$$

for $|z| = r < 1$, then

$$\left|\arg\left(\frac{f(z)}{g(z)} - \beta_1\right)\right| < \frac{\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.8}$$

or

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > \beta_1 \qquad (z \in \mathbb{U}), \tag{2.9}$$

where $\beta_1 = \dfrac{2\beta + \delta k}{2 + \delta k}$ and

$$p(z) = \frac{\dfrac{f(z)}{g(z)} - \beta_1}{1 - \beta_1}.$$

Proof. Note that the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. It follows that

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta\right\}\right| = \left|\arg\left\{(\beta_1 - \beta) + (1-\beta_1)\left(p(z) + \frac{\alpha g(z)}{zg'(z)}zp'(z)\right)\right\}\right|$$

$$< \frac{\pi}{2} + \operatorname{Tan}^{-1}\left(\frac{\delta k|p(z)|}{2\left(\dfrac{2r}{1-r^2} + \dfrac{|\operatorname{Im}\alpha|}{n} + 1\right)}\right)$$

for $|z| = r < 1$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2} \qquad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2},$$

then, by Lemma 2, we have that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell,$$

where

$$\frac{k}{2}\left(a+\frac{1}{a}\right) \leqq \ell \qquad \left(\arg p(z_0) = \frac{\pi}{2}\right)$$

and

$$-\frac{k}{2}\left(a+\frac{1}{a}\right) \geqq \ell \qquad \left(\arg p(z_0) = -\frac{\pi}{2}\right)$$

with $p(z_0) = \pm ia \quad (a>0)$. If $\arg p(z_0) = \dfrac{\pi}{2}$, then it follows that

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\}$$

$$= \arg p(z_0)\left(\frac{\beta_1-\beta}{p(z_0)} + (1-\beta_1)\left(1+\frac{\alpha g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right)\right\}$$

$$= \frac{\pi}{2} + \arg\left\{-\left(\frac{\beta_1-\beta}{a}\right)i + (1-\beta_1)\left(1+i\ell\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right\}$$

$$= \frac{\pi}{2} + \arg I(z_0),$$

where

$$I(z_0) = -\left(\frac{\beta_1-\beta}{a}\right)i + (1-\beta_1)\left(1+i\ell\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right). \tag{2.9}$$

Note that

$$\begin{aligned} \operatorname{Im} I(z_0) &= \frac{\beta-\beta_1}{a} + (1-\beta_1)\ell \operatorname{Re}\frac{\alpha g(z_0)}{z_0 g'(z_0)} \\ &> (1-\beta_1)\delta\ell + \frac{\beta-\beta_1}{a} \\ &\geqq \frac{\delta k}{2}(1-\beta_1)\left(a+\frac{1}{a}\right) + \frac{\beta-\beta_1}{a} \\ &= \frac{\delta k}{2}(1-\beta_1)a > 0 \end{aligned} \tag{2.10}$$

and

$$\operatorname{Re} I(z_0) = (1-\beta_1)\left(1-\ell\operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right) \leqq (1-\beta_1)\left(1+\ell\left|\operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right|\right). \tag{2.11}$$

Letting

$$q(z) = \frac{\alpha g(z)}{zg'(z)} + 1 - \frac{\alpha}{n}, \tag{2.12}$$

we know that $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$. This gives us that

$$|\operatorname{Im} q(z)| = \left| \operatorname{Im} \left(\frac{\alpha g(z)}{zg'(z)} + 1 - \frac{\alpha}{n} \right) \right| \leqq \frac{2r}{1 - r^2} \tag{2.13}$$

for $|z| = r < 1$. Thus we have that

$$\left| \operatorname{Im} \left(\frac{\alpha g(z_0)}{z_0 g'(z_0)} \right) \right| \leqq \frac{2r}{1 - r^2} + \frac{|\operatorname{Im} \alpha|}{n} \qquad (|z| = r < 1). \tag{2.14}$$

Using (2.11) and (2.14), we obtain that

$$\arg I(z_0) = \operatorname{Tan}^{-1} \left(\frac{\operatorname{Im} I(z_0)}{\operatorname{Re} I(z_0)} \right) \geqq \operatorname{Tan}^{-1} \left(\frac{\delta k a}{2 \left(\frac{2r}{1 - r^2} + \frac{|\operatorname{Im} \alpha|}{n} + 1 \right)} \right),$$

which contradicts our condition (2.7).

If $\arg p(z_0) = -\frac{\pi}{2}$, using the same way, we also have that

$$\arg \left\{ (1 - \alpha) \frac{f(z_0)}{g(z_0)} + \alpha \frac{f'(z_0)}{g'(z_0)} - \beta \right\} \leqq - \left\{ \frac{\pi}{2} + \operatorname{Tan}^{-1} \left(\frac{\delta k a}{2 \left(\frac{2r}{1 - r^2} + \frac{|\operatorname{Im} \alpha|}{n} + 1 \right)} \right) \right\},$$

which contradicts (2.7). □

Remark 2 If $f(z)$ satisfies the conditions in Theorem B, then $f(z)$ satisfies the conditions of Theorem 2. In this case, we see that Theorem 2 becomes Theorem B.

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